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DENJOY-TYPE FLOWS ON ORIENTABLE 2-MANIFOLDS OF HIGHER GENUS(1)

BY

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ABSTRACT. The author generalizes A. Denjoy's theory of flows on a torus to compact orientable 2-manifolds of higher genus. Natural extensions of A. Denjoy's hypotheses are made and necessary conditions that a flow satisfy the new hypotheses are given.

I. Intorduction. In 1932, A. Denjoy [1] studied differential equations with no singular points on a 2-torus T satisfying the following hypotheses:

h_1. There is a closed curve Γ, transverse to the flow, which is non-null-homotopic.

h_2. Every trajectory of the flow intersects Γ.

It is well known (i.e. see [2, p. 196]) that every differential equation with no singular points on a torus satisfies h_1. It is not difficult to give examples of such differential equations that do not satisfy hypothesis h_2.

After defining the rotation number α of a flow to be the rotation number of the induced orientation preserving homeomorphism S of Γ onto itself, Denjoy obtained the following results:

Theorem A. *If T^t is a flow on T satisfying hypotheses h_1 and h_2, α is rational if and only if the flow contains a periodic orbit.*

Theorem B. *If T^t is a flow on T of class C^2, satisfying hypotheses h_1 and h_2 and α is irrational, then every trajectory is dense on T.*

In this paper, the author generalizes Denjoy's theory to arbitrary compact orientable 2-manifolds of class C^k, $k \geq 1$.

A flow will be denoted by T^t, T_n will denote a compact orientable 2-manifold of genus n, the positive and negative trajectories with initial point p will be denoted $p(+t)$ and $p(-t)$ respectively and the ω-limit set of $p(+t)$ will be denoted $\Omega(p)$.

In [3], given a flow T^t of class C^k, $k \geq 2$, on T_2, with only stationary points

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of negative index, the author proves the existence of a C^k, closed, non-null-homotopic curve Γ that does not separate T_n and is transverse to the flow. He further conjectures that such a transversal exists for any T_n. Certainly, for any n, it is easy to give examples for which such a transversal exists. Hence, we make the following hypotheses analogous to Denjoy's:

Given a C^k flow T^t on T_n, $k \geq 1$:

H_1. There exists a C^k closed curve Γ transverse to the flow which is non-null-homotopic and does not separate T_n.

H_2. Every trajectory of the flow not a separatrix entering a singular point for positive (negative) time intersects Γ for positive (negative) time.

These two hypotheses will enable the author to show that there is a natural one to one map S' induced by the flow from that subset of Γ lying on trajectories that do not enter singular points of flow for positive time to that subset of Γ lying on trajectories that do not enter singular points of the flow for negative time. It will be clear that S' always has an extension to all of Γ. In fact S' has many extensions to all of Γ, but there exist flows for which no extension S of S' will be orientation preserving. Hence, we make the following hypothesis:

H_3. The flow is one for which an orientation preserving extension S of S' exists.

Definition 1.1. Flows on T_n with a finite number of critical points all of nonzero index satisfying hypotheses H_1, H_2 and H_3 will be called Denjoy-type flows.

Note. S will be seen to be unique for Denjoy-type flows.

Necessary conditions that flows be Denjoy-type will be shown to be that all stationary points be simple singularities of negative index and that each singular point be connected to another (possibly the same one) by at least one trajectory.

The following analgoues to Denjoy's Theorems A and B will be proven.

Theorem A_1. *The rotation number α of a Denjoy-type flow on T_n is rational if and only if the flow contains a periodic orbit or a closed curve consisting entirely of trajectories.*

Theorem B_1. *If the rotation number α of a Denjoy-type flow of class C^2 is irrational and S and S^{-1} are of class C^2, then every trajectory not entering a singular point for positive (negative) time is dense on T_n for positive (negative) time.*

2. Definition of the map S induced by a flow on T_n. The following two lemmas, which will not be used until the proofs of Theorems A_1 and B_1, are stated here as a means of motivating the definition of S and the necessity for hypotheses H_1, H_2 and H_3.

Familiarity with the definition of discrete flow on a closed Jordan curve Γ will be assumed (i.e., see [2, pp. 190–195]). Such a discrete flow will be denoted $\{S^n\}$, where S is the underlying homeomorphism of Γ onto itself. α will denote the rotation number of S. The following two lemmas are well known:

Lemma 2.1. *Let $\{S^n\}$ be a flow on a Jordan curve Γ. Then α is rational if and only if S^k has a fixed point for some $k > 0$.*

Lemma 2.2. *Let $\{S^n\}$ be a flow on a Jordan curve Γ having an irrational rotation number. For any $\gamma \in \Gamma$, let $\Omega(\gamma)$ be the set of limit points of $\{S^k\gamma: k = 0, 1, 2, \cdots\}$. Then $\Omega(\gamma)$ is independent of γ and if S and S^{-1} are of class C^2, $\Omega(\gamma) = \Gamma$.*

The map S we are to define will be one that may serve as the underlying homeomorphism for a discrete flow on the transversal curve Γ of hypothesis H_1. We first prove

Lemma 2.3. *The singular points of any Denjoy-type flow of class C^k, $k \geq 1$, on T_n are simple and of negative index.*

Remark. A simple singular point of negative index will be one which has no elliptic or parabolic sectors. (See [5] for a discussion of these sectors.)

Proof. Let p be a singular point. Either p is a spiral point or small neighborhoods of p can be decomposed into parabolic, elliptic and hyperbolic sectors. If p is a spiral point, hypothesis H_2 is violated, since trajectories entering p are not separatrices and any trajectory close enough to p will enter any small enough neighborhood of p for positive (or negative) time and never leave, thus not intersecting Γ for positive (or negative) time; a contradiction of hypothesis H_2. If p is not a spiral point and p has elliptic or parabolic sectors, in these sectors there are trajectories which are not separatrices that enter any small enough neighborhood of p and do not leave for positive (or negative) time, leading to the same contradiction as before. Hence singular points are simple.

It is well known that

$$2i = 2 + n_e - n_h, \tag{2.1}$$

where i is the index of the singular point, n_e and n_h are the number of elliptic and hyperbolic sectors respectively.

(2.1) implies that the indices of singularities of Denjoy-type flows are simple and of negative index since there are no elliptic sectors or parabolic sectors. Q.E.D.

We now proceed to define an orientation preserving homeomorphism S of Γ onto itself induced by a Denjoy-type flow on T_n.

Let S' be the natural map from that subset of Γ, consisting of points the trajectories through which do not enter singular points before intersecting Γ again, for positive time, to Γ. Specifically, if $p \in \Gamma$ such that $p(+t)$ intersects Γ again. Let t_p be the first nonzero time that $p(+t)$ intersects Γ again. Define $S'p = p(t_p)$. Let Q be the set of points not in the domain of S' and let R be the set of points not in the range of Γ. It is clear that Q consists of those points $q \in \Gamma$ such that $q(+t)$ enters a singular point before intersecting Γ again and R consists of those points $r \in \Gamma$ such that $r(-t)$ enters a singular point before intersecting Γ again. We use the Poincaré-Hopf theorem to determine more about the sets Q and R. (See [4] for a discussion of this theorem.)

Theorem 2.1 (Poincaré-Hopf). *Let M be a C^k, $k \geq 1$, compact manifold with boundary. Let T^t be a C^k flow on M. Then $\Sigma i = \chi(M)$ where Σi is the sum of the indices of the singular points of T^t and $\chi(M)$ is the Euler characteristic of the manifold.*

Comparing this theorem with equation (2.1) and the fact that our singular points have only hyperbolic sectors, we see that the maximum number of trajectories that may enter singularities is $2(2n-2)$ since the Euler characteristic of T_n is $2-2n$. Hence, Q contains at most $2(2n-2)$ points. Suppose $Q = \{q_i\}_{i \leq k}$, $0 \leq k \leq 2(2n-2)$. It is clear that R has the same number of points, $R = \{r_j\}_{j \leq k}$. Order the set Q according to an orientation for Γ.

Given hypothesis H_3, that there is an orientation preserving extension S of S', it is clear how S must be defined. Consider the arc $Q_i = (q_{i-1}, q_i)$ on Γ (take Q_1 to be (q_k, q_1)). Let $R_i = \{r\colon r = p(t_p), p \in Q_i\}$, where t_p is the first time that $p(+t)$ intersects Γ. R_i is also an arc on Γ. S' is orientation preserving on each arc Q_i. The arcs Q_i inherit an ordering on Γ from the ordering of Q. Under any orientation preserving extension S of S' the arcs R_i would inherit the same ordering. Q_i is adjacent to Q_{i+1} for each i and are separated by the point q_i. Then under any choice for S, R_i is adjacent to R_{i+1} and is separated by one point of R, call it r_i. Then for any orientation preserving extension S of S', Sq_i must be defined to be r_i. S is readily seen to be one to one and onto. The continuity of S and S^{-1} follow easily from the continuous dependence of the trajectories of the flow on initial conditions and hypothesis H_2. Hence S is an orientation preserving homeomorphism of Γ onto itself and is uniquely determined by the flow.

The following example (Diagram 2.1) of a flow on T^2 illustrates an S' that has no orientation preserving extension; hence the necessity of hypothesis H_3. In this example, T_2 is obtained by identifying opposite sides of the rectangle with the same orientation and identifying the boundaries of the discs with opposite orientation; p_1 and p_2 represent two simple saddle points.

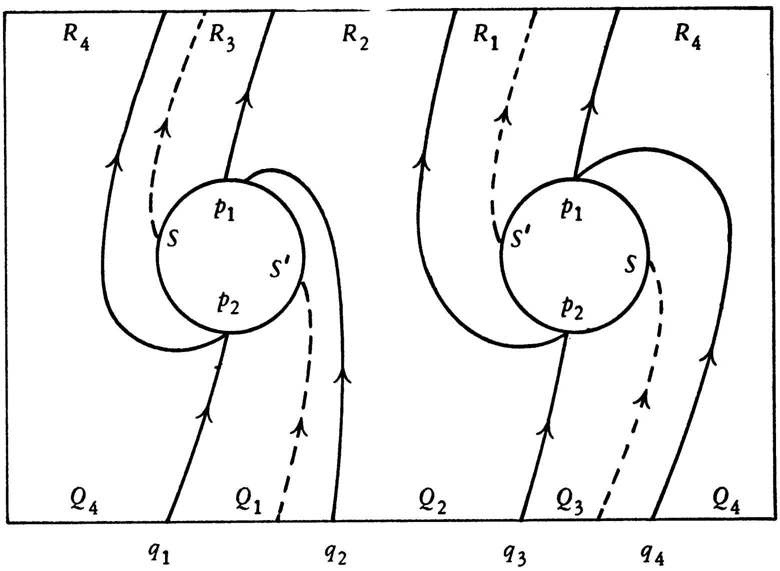

Diagram 2.1

and identifying the boundaries of the discs with opposite orientation; p_1 and p_2 represent two simple saddle points.

3. **Necessary conditions that a flow T^t on T^n be Denjoy-type.** Let T^t be a Denjoy-type flow of class C^k, $k \geq 1$. Let $O_i = \{p: p = q(t)$ for some $q \in Q_i$, $0 < t < t_q\}$, where t_q is the first time such that $q(t)$ intersects Γ again. By hypothesis H_2, the O_i's taken together with Γ, the singular points of T^t and the trajectories that enter singular points for positive or negative time constitute all of T_n. Hence the boundary of O_i consists of arcs of Γ, singular points and arcs of trajectories that enter singular points. The following lemma describes O_i more precisely.

Lemma 3.1. *O_i is the C^k diffeomorphic to $I \times I$, where $I = (0, 1)$.*

Proof. Let g be a diffeomorphism between I and Q_i. Define a function $f: I \times I \to O_i$ by

$$f(x, s) = p(xt_q, g(s)),$$

where $q = g(s)$. That is, $f(x, s)$ is the point in O_i on the trajectory through $g(s)$ at time xt_q. f is clearly one to one and onto. The C^k character of t_q with respect to q and of the trajectories of the flow with respect to initial conditions easily implies the C^k differentiability of f and f^{-1}. Q.E.D.

Note. If $Q_i = \Gamma$, then O_i is diffeomorphic to the circular cylinder $I \times C$.

Lemma 3.2. *If T^t is a Denjoy-type flow of class C^k, $k \geq 1$, there must be at least one point $q \in \Gamma$ such that $q(+t)$ enters a singular point without intersecting Γ again. (Hence there is also one point $r \in \Gamma$ such that $r(-t)$ enters a singular point without intersecting Γ again.)*

Proof. Suppose there is no point q such that $q(+t)$ enters a singular point without intersecting Γ again. Then there is only one set Q_i as above, call it Q, and $Q = \Gamma$. The corresponding set $O = \{p: p = q(t), q \in \Gamma, 0 < t < t_q\}$ is a circular cylinder whose closure is given by cl $O = \{p: p = q(t), q \in \Gamma, 0 \leq t \leq t_q\}$. It is clear that cl O is a torus. Hence, there is a torus embedded as a close submanifold of T_n, a contradiction. Q.E.D.

Lemmas 3.1 and 3.2 are fundamental in giving us a workable description of T_n. T_n can be thought of as $(\bigcup_i O_i) \cup (\bigcup_i \partial O_i)$. We already know that ∂O_i consists of arcs of trajectories through points of Γ entering singular points without intersecting Γ again (i.e., trajectories $q_i(+t)$, $r_i(-t)$, singular points, trajectories connecting singular points and arcs of Γ). We also know that any such object is in ∂O_i. It is also clear that a portion of any separatrix entering a singular point will appear on some of the ∂O_i's. T_n is now decomposed into closed strips $\bar{O}_i$ where the boundaries of these strips are to be identified suitably to yield T_n. A more careful examination of O_i will show that one pair of opposite boundary arcs of O_i consists of arcs of Γ and the singular points and trajectories entering singular points lie on the other pair of opposite boundary arcs. Also, the only points that lie on both pairs of opposite boundary arcs are the points $q_j \in Q$ and $r_j \in R$. Each q_j and each r_j will have exactly two representations in the totality of strips O_i, namely on one on O_{j-1} and one on O_j. Likewise, the trajectories $q_i(+t)$ and $r_j(-t)$ will have exactly two representations, one on O_{j-1} and one on O_j. The singular points that these trajectories enter will have at least two representations one of which is on O_{j-1} and another on O_j. Trajectories connecting singular points will have exactly two representations on the boundaries ∂O_i which are to be identified. If there were three or more representations of the same trajectory, upon identifying them, the result cannot be a 2-manifold since any neighborhood of a point on such a trajectory would not be homeomorphic to an open set in R^2 (see Diagram 3.1).

Theorem 3.1. *Necessary conditions that a flow be Denjoy-type on T_n of class C^k, $k \geq 1$, are that all singular points be simple of negative index and that each singular point be connected to another (possibly the same one) by at least one trajectory.*

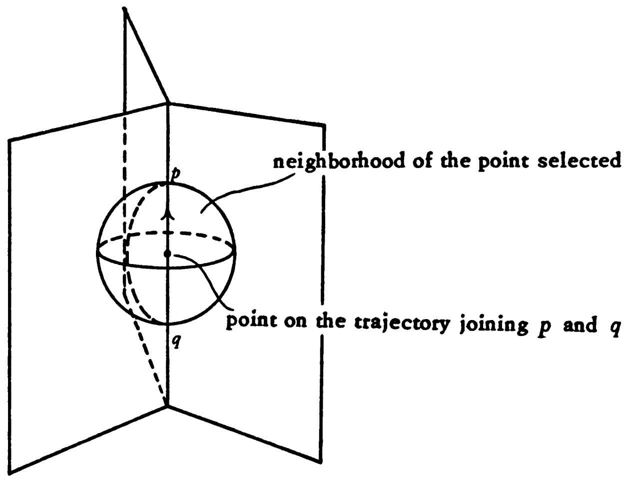

Diagram 3.1

Proof. To prove this theorem, it is only necessary to prove that given any singular point, there is at least one trajectory connecting this singular point to another (possibly the same one). The remaining part of this theorem is covered by Lemma 2.3.

Consider the two edges of the ∂O_i's bounded by the points q_i and r_i. If either one of these edges contains more than one representation of a singular point the theorem is proven, for then there is a segment of an edge between the two singular points which represents a trajectory connecting them. If the both edges contain exactly one representation of a singular point it must be the same singular point p, since we must be able to identify the representations of the trajectories $q_i(+t)$ and $r_i(-t)$ that appear on the two edges. Hence both edges bounded by q_i and r_i are completely identified. If these are the only representations of p, then p has exactly two separatrices entering it, one for positive time and one for negative time. This implies that there are only two hyperbolic sectors associated with p, i.e., p is a singularity of index zero (see Formula 2.1). There are, thus, other representations of p. But identifying all these representations of p leads to an object that cannot be a 2-manifold for then p does not have a neighborhood homeomorphic to an open set in R^2. Q.E.D.

4. **Analysis of Denjoy-type flows.** In § 2, we saw how a Denjoy-type flow T^t on T_n induced a discrete flow $\{S^n\}$ on the transversal Γ of hypothesis H_1. The rotation number α of this discrete flow was taken to be the rotation number

of the flow T^t. We prove the Theorems A_1 and B_1 as stated in the introduction.

Theorem A_1. *The rotation number α of a Denjoy-type flow on T_n is rational if and only if the flow contains a periodic orbit or a closed curve consisting entirely of trajectories.*

Proof. The induced map S of § 2 is an orientation preserving homeomorphism of Γ. Suppose α is rational. By Lemma 2.1, S^k has a fixed point for some $k > 0$. Call the fixed point γ_0. If $\gamma_0(+t)$ does not enter a singular point, then by the way we defined S, $\gamma_0(+t)$ must be a periodic orbit which intersects Γ k times. If $\gamma_0(+t)$ enters a singular point, let q be the last point on Γ through which $\gamma_0(+t)$ goes before entering the singular point. Consider the point $r = Sq$. We know that q is connected to r by paths of trajectories and singular points, also by the way we defined S. Consider $r(+t)$. $r(+t)$ may or may not enter a singular point. If it does, carry out the process again. This process can be carried out a most finite number of times, since k is finite. At any rate, after having intersected Γ k times, we are back to γ_0. This completes the proof of the necessary condition; the proof of the sufficient condition is clear. Q.E.D.

Theorem B_1. *If the rotation number α of a Denjoy-type flow of class C^2 is irrational and S and S^{-1} are of class C^2, then every trajectory not entering a singular point for positive (negative) time is dense on T_n for positive (negative) time.*

Proof. Since S and S^{-1} are C^2 and α is irrational, the set of points $S^n\gamma$, $n = 0, 1, 2, \cdots$, is dense on Γ, by Lemma 2.2. Let $p(+t)$ be a semiorbit that does not enter a singular point. Suppose $p(+t)$ is not dense on T_n. Then there is a neighborhood N on T_n which $p(+t)$ does not intersect. By hypothesis H_1, $p(+t)$ intersects Γ and hence $p(+t)\cap\Gamma$ is dense on Γ. Now let $p_0 \in N$ such that $p_0(+t)$ does not enter a singular point. Then $p_0(+t)$ intersects Γ at a point q_0 at time t_{p_0}. By continuity of the differential equation, we can choose an arc segment l on Γ, containing q_0, which is so small that $p_q = q(t_{p_0})$ is contained in N for all $q \in l$. Since l is the image under trajectories of points p_q in N and N contains no points of $p(+t)$, $p(+t)$ cannot intersect l. This contradicts the fact that $p(+t) \cap \Gamma$ is dense on Γ. Q.E.D.

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